Resolution of Bourgain's hyperplane conjecture

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based on a joint work with Bo'az Klartag

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In the mid 80s, Bourgain came up with the following innocent looking question:

Slicing (or Hyperplane) conjecture

Let K be a convex body in $\mathbb{R}^n$ of volume 1. Is it true that there exists an affine hyperplane H such that

$$\mathrm{vol}_{n-1}(K \cap H) \geq c,$$

where $c > 0$ is a universal constant?

This remained open for 40 years. In a recent joint work with Bo'az Klartag we finally solved it, positively.

The hyperplane conjecture is related to the following.

Busemann-Petty problem (1950s)

Suppose K, L are symmetric ($K = -K$) convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H)$$

for any hyperplane H passing through the origin. Does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

The answers turns out to be negative in high dimensions. In fact:

- Keith Ball 1986: Central sections of the hypercube of volume 1 have volume $\leq \sqrt{2}$ in any dimension. On the other hand the volume of central sections of the Euclidean ball of volume 1 converges to $\sqrt{\mathrm{e}}$ when $n \to +\infty$.
- The exact answer to Busemann-Petty is yes in dimension ≤ 4 and no in dimension ≥ 5 but this is another story.

The hyperplane conjecture is actually equivalent to the following:

Corrected Busemann-Petty

Let K, L be centrally symmetric and such that

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H)$$

for any hyperplane H passing through the origin. Does it follow that

$$\mathrm{vol}_n(K) \leq C \cdot \mathrm{vol}_n(L),$$

where C is a universal constant?

If K is a convex body we consider its barycenter and inertia matrix, namely the expectation and the covariance of a random vector X uniformly distributed on K. Thus $\mathrm{bar}(K) := \mathbb{E}X$ and

$$\mathrm{cov}(K) := (\mathbb{E}X \otimes X) - (\mathbb{E}X) \otimes (\mathbb{E}X).$$

Isotropic position

We say that K is isotropic position if

$$\mathrm{bar}(K) = 0 \quad \text{and} \quad \mathrm{cov}(K) = \mathrm{Id}_{\mathbb{R}^n}.$$

Any convex body admits an affine image that is isotropic.

It follows from classical Brunn-Minkowski theory that when K is in isotropic position then all sections of K through the origin have volume of order 1.

The question is the order of magnitude of $\mathrm{vol}_n(K)$.

Isotropic constant

If K is a convex body in $\mathbb{R}^n$, define the isotropic constant of K by

$$L_K := \frac{\det \operatorname{cov}(K)^{1/2n}}{\operatorname{vol}_n(K)^{1/n}}$$

This is an affine invariant quantity: $L_K := \operatorname{vol}_n(\widetilde{K})^{-1/n}$ where $\widetilde{K}$ is the affine image of K that is isotropic.

It is an exercise to see that this is minimized by ellispsoids and that $L_{\mathcal{E}} \sim (2\pi \mathrm{e})^{-1/2}$ for ellipsoids.

The question is the upper bound:

Bourgain's slicing again

Is it true that $L_K \leq C$ for any convex body in $\mathbb{R}^n$, and where C is a universal constant?

Log-concave measures

A measure μ on $\mathbb{R}^n$ of the form

$$\mu(dx) = \mathrm{e}^{-V(x)}\, dx$$

with $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex is called log-concave.

This class contains the class of uniform measures on convex sets, it also contains Gaussian measures, or Gaussian measures conditioned on some convex set, ...

A nice thing about this class is that it is stable under various operations such as taking products and marginals. In particular the convolution of two log-concave measures is again log-concave.

Entropy

Given a random vector X on $\mathbb{R}^n$ with absolutely continuous law μ, the Shannon entropy of μ (or X) is

$$\mathrm{Ent}(X) = \mathrm{Ent}(\mu) = -\int_{\mathbb{R}^n} f \log f \, dx = -\mathbb{E} \log f(X)$$

where f is the density of μ.

Note: If X is uniform on a set K then $\mathrm{Ent}(X) = \log \mathrm{vol}_n(K)$.

Isotropic constant of a probability measure on $\mathbb{R}^n$

$$L_X = L_\mu = \frac{\det \mathrm{cov}(\mu)^{1/(2n)}}{\left(\mathrm{e}^{\mathrm{Ent}(\mu)}\right)^{1/n}}$$

Again this is affine invariant: $L_\mu = \exp(-\mathrm{Ent}(\widetilde{\mu})/n)$ where $\widetilde{\mu}$ is the affine image of μ that is isotropic.

It is well-known that among measures with prescribed covariance the Gaussian measure maximizes the entropy.
In other words

$$L_\mu \geq L_{\mathrm{Gaussian}} = \frac{1}{\sqrt{2\pi \mathrm{e}}}, \quad \forall \mu.$$

There is no upper bound for general μ, but what if we impose log-concavity?

Slicing conjecture again

Is it true that $L_\mu \leq C$ when μ is a log-concave probability measure on $\mathbb{R}^n$?

We can also reformulate as follows: Is it true that $(1/n)$ times the entropy of an isotropic log-concave probability measure in dimension n is of constant order?

Let $L_n = \sup\{L_K, K \text{ convex body in } \mathbb{R}^n\}$ (Bourgain constant)
The bound $L_n = O(\sqrt{n})$ is trivial.

1. Bourgain 1984 proved $L_n = O(n^{1/4} \log n)$ using Gaussian processes
2. Klartag removed the log in 2006 (by a different method)
3. Lee-Vempala 2016 gave another proof of $L_n = O(n^{1/4})$ using Eldan's stochastic localization.
4. Yuansi Chen's breakthrough from 2020: $L_n = n^{o(1)}$, some new way of controlling things in stochastic localization
5. Klartag, L. 2022: $L_n = O(\log^4 n)$, stochastic localization + some tools from spectral theory, improved by Jambulapathi, Lee, Vempala: $L_n = O(\log^{2.223\ldots} n)$.
6. Klartag 2023: $L_n = O(\sqrt{\log n})$. Stochastic localization + improved Lichnerowicz
7. Guan 2024: $L_n = O(\log \log n)$: method from KL 2022, but with a new key estimate in stochastic localization.

It turns out that the key estimate in Guan's work (to which I'll come back later on) was the missing link in an approach that we had with Bo'az for slicing.
So we can now prove the following.

Theorem [Klartag, L. 2024]

$$L_n = O(1).$$

The starting point for our proof is an approach suggested first by Keith Ball.

Shannon-Stam's inequality

Suppose X is a random vector and Y is an independent copy of X, then

$$\mathrm{Ent}\left(\frac{X+Y}{\sqrt{2}}\right) \geq \mathrm{Ent}(X).$$

There's equality in Shannon-Stam iff X is a Gaussian vector.

Definition: Shannon-Stam deficit

$$\Delta(X) := \mathrm{Ent}\left(\frac{X+Y}{\sqrt{2}}\right) - \mathrm{Ent}(X).$$

A stability estimate for Shannon-Stam is a lower bound for $\Delta(X)$ in terms of some distance from X to the set of Gaussian vectors.

Fact [Ball, Nguyen 2013]

When X is log-concave on $\mathbb{R}^n$ we have

$$\Delta(X) \leq Cn,$$

where C is a universal constant.

As a result if we could lower bound the deficit by (say) the entropy gap between X and the Gaussian vector having the same covariance as X then we would nail slicing.
This allowed Ball and Nguyen to prove that if X is isotropic and log-concave then

$$L_X \leq \exp(C' \cdot C_P(X))$$

where $C_P(X)$ is the Poincaré constant of X
This is not good enough to get any meaningful bound on L_n but it does show that slicing is implied by the Kannan, Lovasz and Simonovits conjecture. (We'll come back to that).

The classical proof of the Shannon-Stam inequality uses the heat flow.

Let $X_s = X + B_s$ and where (B_s) be a standard Brownian motion on $\mathbb{R}^n$, independent of X.

- This preserves log-concavity: if X is log-concave then X_s too is log-concave
- $\mathrm{cov}(X_s) = \mathrm{cov}(X) + s\mathrm{Id}_{\mathbb{R}^n}$.
- Differentiating the entropy along the heat-flow gives a tractable formula (de Bruijn's identity).

However, sticking to this classical framework doesn't lead to good enough stability estimates for our purposes. Instead we use a variant that is called stochastic localization.

In a nutshell, Eldan's stochastic localization (2012), or rather its simplified version by Lee-Vempala (2015) relies on two things:

- Time reversal: we set $t = 1/s$
- Conditioning: We let μ_t be the law of X conditioned on X_s. Note that μ_t is a random measure.

This defines some sort of interpolation:

- when $t \to 0$, which corresponds to $s \to +\infty$, X and X_s become independent so $\mu_t \to \mu$.
- When $t \to \infty$, we have $s \to 0$ and $X_s \to X$, and μ_t converges to a Dirac point mass at $X \sim \mu$.

Hence the name stochastic localization.
The point of the time reversal is that we can then come up with some sort of stochastic differential equation for μ_t which allows to compute things using Itô calculus. (I won't enter into the details here)

Let A_t be the covariance of μ_t. Note that μ_t is a random measure, so A_t is a random matrix. Some properties:

- $\frac{d}{dt}\mathbb{E}A_t = -\mathbb{E}A_t^2$.
- When μ is isotropic we have $\mathbb{E}A_t \leq (1+t)^{-1}\mathrm{Id}$
- When μ is log-concave we have $A_t \leq t^{-1}\mathrm{Id}$, a.s.
- Formulas for the entropy: when μ is isotropic

$$\begin{aligned}\mathrm{Ent}(\gamma) - \mathrm{Ent}(\mu) &= \frac{1}{2}\int_0^\infty \mathrm{Tr}\left(\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right) dt \\ &= \frac{1}{2}\int_0^\infty (1+t)\cdot \mathrm{Tr}\,\mathbb{E}\left[\left(\frac{\mathrm{Id}}{1+t} - A_t\right)^2\right] dt.\end{aligned}$$

Here γ is the standard Gaussian measure.
This is essentially a reformulation of de Bruijn's identity + some integration by parts.

Elaborating on a proof of Shannon-Stamm using some stochastic calculus from Lehec (2012), Eldan and Mikulincer (2017) proved that in the isotropic log-concave case the deficit satisfies

$$\Delta(\mu) \geq c \int_0^\infty (1+t)\mathrm{Tr}\left[\mathbb{E}(A_t - \mathbb{E}A_t)^2\right] dt.$$

Recall also Ball-Nguyen: $\Delta(\mu) \leq Cn$.
Plugging back into the second formula for the entropy yields

$$\begin{aligned}
\mathrm{Ent}(\gamma) - \mathrm{Ent}(\mu) &= \frac{1}{2}\int_0^\infty (1+t)\cdot \mathrm{Tr}\,\mathbb{E}\left[\left(\frac{\mathrm{Id}}{1+t} - A_t\right)^2\right] dt \\
&\leq \frac{1}{2}\int_0^\infty (1+t)\cdot \mathrm{Tr}\left[\left(\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right)^2\right] + C'n.
\end{aligned}$$

Guan (2024) proved the following.

Theorem

When μ is isotropic and log-concave

$$\mathbb{E}\mathrm{Tr}A_t^2 \leq Cn, \quad \forall t > 0.$$

This relies on a series of works on stochastic localization (Eldan, Lee, Vempala, Chen, Klartag, Lehec), but there was quite a gap between what was known before and this statement by Guan, which is a real tour de force.

Since $A_0 = \mathrm{cov}(\mu) = \mathrm{Id}$ and $\frac{d}{dt}\mathbb{E}\mathrm{Tr}A_t = -\mathbb{E}\mathrm{Tr}A_t^2$ this implies

$$\mathbb{E}\mathrm{Tr}A_t \geq (1 - Ct)n \quad \forall t$$

In particular $t_0 := 1/(2C)$ satisfies

$$\mathbb{E}\mathrm{Tr}A_{t_0} \geq \frac{n}{2}.$$

Unfortunately the inequality $\mathbb{E}\mathrm{Tr}A_{t_0} \geq n/2$ is not good enough for our needs. We would be better off with

$$\mathbb{E}A_{t_0} \geq c \cdot \mathrm{Id}.$$

In other words we want a lower bound on all eigenvalues of $\mathbb{E}A_{t_0}$, and not just their sum.
This can be done by passing to a subspace. Indeed:

- $\mathbb{E}A_t$ is non increasing, so we also have $\mathbb{E}A_{t_0} \leq A_0 = \mathrm{Id}$. So all eigenvalues are $\mathbb{E}A_{t_0}$ are less than 1.
- Since their sum is at least $n/2$, at least a third of them should be $\geq 1/4$.
- Thus there exists a subspace E of dimension $\geq n/3$ such that

$$p_E\left(\mathbb{E}A_{t_0}\right)(p_E)^* \geq \frac{1}{4}\mathrm{Id}_E.$$

  where $p_E \colon \mathbb{R}^n \to E$ is the orthogonal projection onto E.

Why can we afford to pass to a subspace?

- First of all from $A_t = \mathrm{cov}(X \mid X_s)$ we get

$$\frac{1}{4}\mathrm{Id}_E \leq p_E \left(\mathbb{E}A_{t_0}\right)(p_E)^* \leq \mathbb{E}A_{t_0,E}.$$

where $A_{t,E}$ is the covariance of the stochastic localization associated to the marginal $\mu_E := p_E\#\mu$.

- Moreover, using the notion of M-ellipsoids, Bourgain, Klartag and Milman (2003) (essentially) showed that if K is the worst isotropic convex body: namely if

$$L_K = L_n$$

and if μ is the uniform measure on K, then for every subspace E of dimension proportional to n we have

$$L_n \leq C \cdot L_{\mu_E}$$

So it is enough to bound the isotropic constant of one particular marginal to get slicing.

Thus it is enough to prove that an isotropic log-concave measure for which $\mathbb{E}A_{t_0} \geq c_0 \mathrm{Id}$ for some $t_0 \geq c_1$ has bounded isotropic constant. (c_0, c_1 are universal constants).

- Using $(d/dt)\mathbb{E}A_t = -\mathbb{E}A_t^2$ and $A_t \leq t^{-1}\mathrm{Id}$ a.s. we then get after some elementary analysis

$$\mathbb{E}A_t \geq \frac{c_2}{1+t}\mathrm{Id}, \quad \forall t > 0.$$

- We plug this into the formula for the entropy

$$\begin{aligned}
\mathrm{Ent}(\gamma) - \mathrm{Ent}(\mu) &= \frac{1}{2}\int_0^\infty \mathrm{Tr}\left(\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right) dt \\
&\leq \frac{1}{2}\int_0^\infty (1+t)\mathrm{Tr}\left[\left(\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right)^2\right] + Cn \\
&\leq (1-c_2)(\mathrm{Ent}(\gamma) - \mathrm{Ent}(\mu)) + Cn.
\end{aligned}$$

- Thus $\mathrm{Ent}(\gamma) - \mathrm{Ent}(\mu) \leq (C/c_2)n$, hence $L_\mu \leq C'$, which is the result.

Very recently Pierre Bizeul has come up with an alternate proof. His proof also relies on stochastic localisation and Guan's bound. However it avoids all of the information theoretic part of our argument. Instead, it is based on some reformulation of slicing in terms of small ball probability estimates (due to Paouris and Dafnis (2009)). The later relies on M-ellipsoid theory. All in all Bizeul's proof is not dramatically different from ours, but arguably a bit simpler.

The slicing problem is related to the Kannan, Lovasz, Simonovits conjecture from 1995 and to the thin-shell conjecture.

- Poincaré constant of μ: Best constant $C_P(\mu)$ in

$$\mathrm{var}_\mu(f) \le C_P(\mu) \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu.$$

- KLS constant:

$$\psi_n^2 = \sup\{C_P(\mu) \colon \mu \text{ isotropic and log-concave in } \mathbb{R}^n\}.$$

- KLS conjecture:

$$\psi_n = O(1)$$

  Their original motivation was related to the computational complexity of certain randomized algorithms.

- Thin-Shell constant

$$\sigma_n^2 := \sup\left\{\frac{1}{n}\mathrm{var}_\mu(|x|^2) \colon \mu \text{ isotropic and log-concave in } \mathbb{R}^n\right\}.$$

- Thin-Shell conjecture: Antilla, Ball, Perissinaki (2003) Bobkov, Koldobski (2003)

$$\sigma_n = O(1)$$

This is related to the Central limit problem for convex sets of Sudakov (1976) and Diaconis - Freedman (1984).

- Hierarchy: Obviously $\sigma_n \le 2\psi_n$ and Eldan and Klartag (2011) proved that $L_n \le C\sigma_n$. Thus

$$L_n \le C\sigma_n \le 2C\psi_n.$$

In terms of the conjectures

KLS $\Rightarrow$ Thin-Shell $\Rightarrow$ Slicing

- As of today the best known bound on the KLS constant is

$$\psi_n = O(\sqrt{\log n})$$

  from Klartag (2023). Until a few months ago this was also the best bound on L_n and σ_n.
- Guan proved

$$\sigma_n = O(\log \log n)$$

  and we proved

$$L_n = O(1).$$

- What's next? Maybe Guan's breakthrough can unlock thin-shell too. For KLS however, there are some serious obstructions with the current approach.